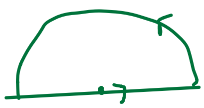
Example (I stole this from the wikipedia page on the Residue Theorem. Also relates to your homework problem 4.3.17) These integrals arise in probability theory: Show that for $b \geq 0$,

$$\int_{-\infty}^{\infty} \frac{\cos(b\,x)}{x^2+1}\,dx = \int_{-\infty}^{\infty} \frac{\overset{\cos bx + i\sin bx}{e^{i\,b\,x}}}{x^2+1}\,dx = \pi\, e^{-b}$$

First check that the method two pages back fails for the function you might try first,

- $f(z) = \dfrac{\cos(b\,z)}{z^2+1}$, when $b > 0$....in both the upper and lower half plane.



$$\cos b(x+iy) = \cos(bx+iby) = \frac{1}{2}\left(e^{i(bx+iby)} + e^{-i(bx+iby)}\right)$$

$e^{ibx}e^{-by}$ good in U.H.P bad in L.H.P.

$e^{-ibx}e^{by}$ bad in UHP good in LHP.

$$\left|e^{ib(x+iy)}\right| = \left|e^{ibx}e^{-by}\right| \leq e^{-by} \leq 1$$

in U.H.P.

So thm will apply!

to be continued!!

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① $$\int_{-\infty}^{\infty} \frac{\cos(b\,x)}{x^2+1}\,dx = \int_{-\infty}^{\infty} \frac{\overset{\cos bx + i\sin bx}{e^{i\,b\,x}}}{x^2+1}\,dx = \pi\, e^{-b}$$

Wednesday: $f(z) = \dfrac{e^{ibz}}{z^2+1}$.

① $$\int_{-R}^{R} \frac{\cos bx}{x^2+1} + i\underbrace{\int_{-R}^{R}\frac{\sin bx}{x^2+1}}_{\int_{-R}^{R}\mathrm{odd}(x)dx \,=\, 0.}\,dx$$

$\dfrac{\sin bx}{x^2+1}\ \dfrac{\text{odd}}{\text{even}} = \text{odd fcn}$

$\lim_{R\to\infty}$ yields ①

② justify contour procedure.

$$|f(z)| = \left|\frac{e^{ib(x+iy)}}{z^2+1}\right| \leq \frac{e^{-by}}{|z|^2-1} \leq \frac{1}{|z|^2-1}$$

in UHP

gives quadratic decay in UHP.

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\left(\sum \text{residues in UHP}\right)$$

$$f(z) = \frac{e^{ibz}}{z^2+1} = \frac{e^{ibz}}{(z-i)(z+i)}$$

simple pole @ $z = i$

$$\mathrm{Res}(f;i) = \lim_{z\to i}\frac{e^{ibz}}{z+i} = \frac{e^{-b}}{2i}$$

$$\therefore \int_{-\infty}^{\infty}\frac{\cos bx}{x^2+1}\,dx = 2\pi i\left(\frac{e^{-b}}{2i}\right) = \boxed{\pi e^{-b}}$$

Math 4200
Wednesday November 18
4.3 Integral applications of the residue theorem; we'll discuss 4.4 magic formulas for series and for certain infinite sums of analytic functions on Friday.

Announcements: We'll finish the last example on Monday, do a few more in today's notes before your quiz. - it's a HW problem.

reminder: HW for Friday November 20
4.3: 1, 2, 4, 7, 10, 14, 17, 20ab. (I've swapped #7 for #6.)
There are a lot of good worked examples in the text. We discuss related examples today and in Monday's notes.

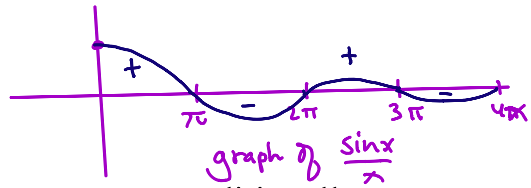
Example (Relates to homework problem 4.3.2). Show

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \int_0^{\infty} \frac{\sin(x)}{x}\,dx = \frac{\pi}{2},$$

Integral does not converge absolutely as $R \to \infty$. Does converge conditionally by alternating series test

using

$$\int_{\gamma_{\varepsilon,R}} \frac{e^{iz}}{z}\,dz$$



graph of $\frac{\sin x}{x}$

Note, this improper integral does not converge absolutely, but converges conditionally by the alternating series test....and also, we use an interesting contour and "principal value" techniques to evaluate it.

- $\Gamma = \gamma_1 + \gamma_\varepsilon + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5$

γ_1: $\displaystyle\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx = \int_{-R}^{-\varepsilon} \frac{\cos x}{x}\,dx + i\int_{-R}^{-\varepsilon} \frac{\sin x}{x}\,dx$

$z = x$
$dz = dx$

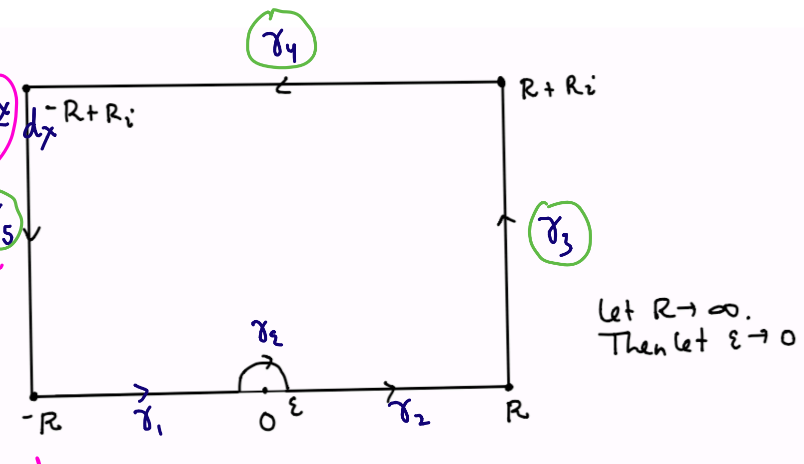
cancel!

odd

γ_2: $\displaystyle\int_{\varepsilon}^{R} \frac{\cos x}{x}\,dx + i\int_{\varepsilon}^{R} \frac{\sin x}{x}\,dx$

odd integrand

even integrand



Let $R \to \infty$. Then let $\varepsilon \to 0$

$$\int_{\gamma_1+\gamma_2} \frac{e^{ix}}{x}\,dx = 2i\int_{\varepsilon}^{R} \frac{\sin x}{x}\,dx$$

$$\int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,dz = \underbrace{\int_{\gamma_\varepsilon} \frac{1}{z}\,dz}_{z=\varepsilon e^{i\theta}} + \underbrace{\int_{\gamma_\varepsilon} \frac{e^{iz}-1}{z}\,dz}_{\text{removable sing at } 0}$$

$$= -\int_0^{\pi} \frac{1}{\cancel{\varepsilon}\cancel{e^{i\theta}}}\, i\cancel{\varepsilon}\cancel{e^{i\theta}}\,d\theta$$

$= -i\pi$

makes sense

so integrand is bounded near origin, by "M"

$$\left|\int_{\gamma_\varepsilon} -\right| \le M\pi\varepsilon$$

$\downarrow$ as $\varepsilon \to 0$.

0

$$\boxed{\text{Claim: } \int_{\gamma_3+\gamma_4+\gamma_5} |f(z)|\,|dz| \to 0 \quad \text{as } R \to \infty.}$$

$\frac{e^{iz}}{z}$ no sings inside Γ

If true.

fixed R, ε

$$\int_{\Gamma} f(z)\,dz = 2\pi i\,(\,0\,)$$

$$\Rightarrow \int_{\gamma_1+\gamma_2} \frac{e^{iz}}{z}\,dz + \int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,dz + \int_{\gamma_3+\gamma_4+\gamma_5} \frac{e^{iz}}{z}\,dz = 0$$

e.g. $\displaystyle\int_{\gamma_3} |f(z)|\,|dz|$.

$$\left|\frac{e^{i(R+iy)}}{|R+iy|}\right| \le \frac{e^{-y}}{R}.$$

$$\Vert\ \int_0^R \frac{e^{-y}}{R}\,dy$$

$$= \frac{1}{R}\left.\frac{e^{-y}}{-1}\right]_0^R =$$

$\lim_{R\to\infty}$: $\displaystyle 2i\int_{\varepsilon}^{\infty} \frac{\sin x}{x}\,dx - (i\pi + \text{error}) + 0$

$\lim_{\varepsilon \to 0}$: $2iI - i\pi = 0 \Rightarrow I = \boxed{\frac{\pi}{2}}$

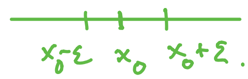
$\frac{1}{R}\left[-e^{-R}+1\right] \le \frac{1}{R}$

In the previous exercise $\frac{e^{iz}}{z}$ has a singularity at $z=0$ even though $\frac{\sin(x)}{x}$ is continuous at $x=0$. There is a general class of integrals, called *Principal Value* (or *PV*) integrals, that one can compute, even when the actual integral doesn't exist. These PV integrals are often important in e.g. physics, I think.

<u>Def</u> If f is continous on $[a, b]$ except at $x_0 \in (a, b)$ then

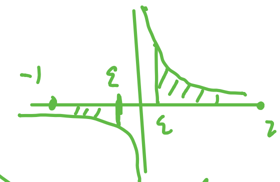
$$PV\left(\int_a^b f(x)\,dx\right) := \lim_{\varepsilon \to 0}\left(\int_a^{x_0-\varepsilon} f(x)\,dx + \int_{x_0+\varepsilon}^b f(x)\,dx\right)$$

provided the limit exists.



<u>Example</u>

$$PV\left(\int_{-1}^2 \frac{1}{x}\,dx\right) = \ln(2)$$



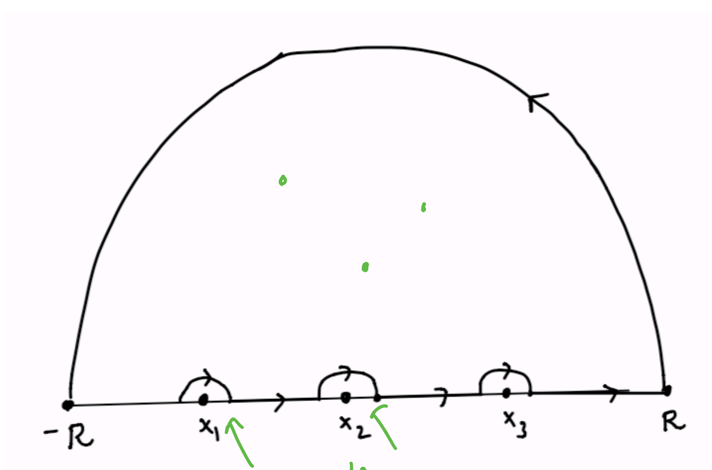
lim

$\int_{-1}^{-\varepsilon} \frac{1}{x} + \int_{\varepsilon}^{2} \frac{1}{x}$

even though

$$\int_{-1}^0 \frac{1}{x}\,dx = -\infty, \quad \int_0^2 \frac{1}{x}\,dx = +\infty\,.$$

Using principal value ideas one can often compute $PV\left(\int_{-\infty}^{\infty} f(x)\,dx\right)$ using contours like the one below. This is Proposition 4.3.11 in the text, of which our worked example was an instance.



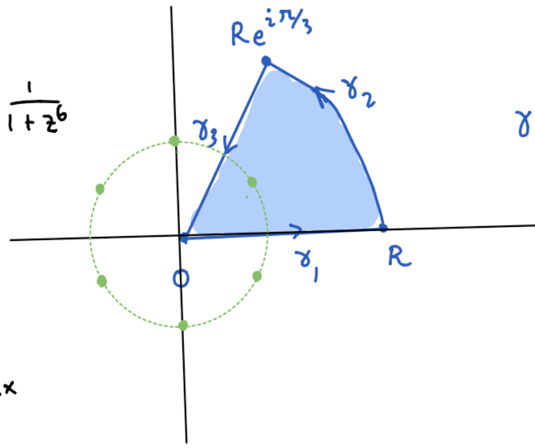
simple poles.

have a look at these!

Suggested contour for 4.3.4 (see worked example 4.3.20 in text)

4.3.4 $\displaystyle\int_0^\infty \frac{1}{1+x^6}\,dx = \frac{\pi}{3}$

$f(z) = \dfrac{1}{1+z^6}$



$\gamma = \gamma_1 + \gamma_2 + \gamma_3$

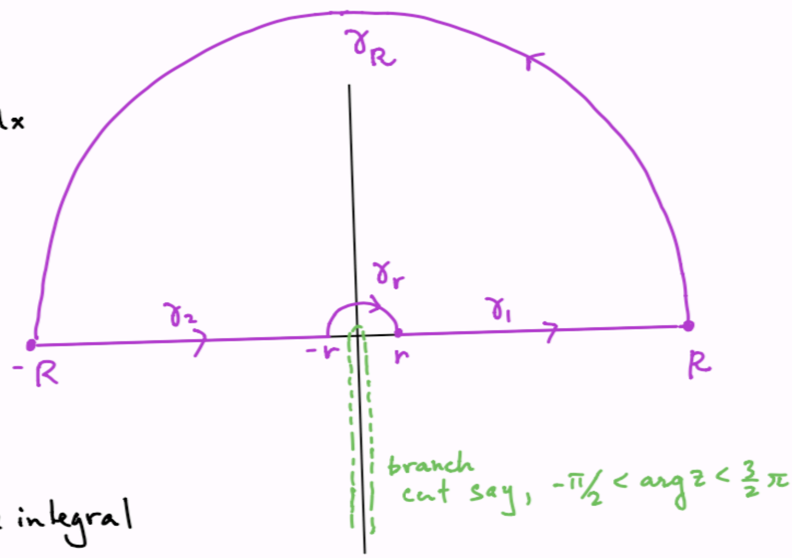
(using $\frac{1}{2}\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^6}\,dx$ means 3 singularities in upper half plane, messy algebra)

- key point: $\displaystyle\int_{\gamma_3} \frac{1}{1+z^6}\,dz = -\int_0^R \frac{1}{1+x^6}\,e^{i\pi/3}\,dx$

$z = e^{i\pi/3}x$

Suggested contour:

4.3.14 $\displaystyle\int_0^\infty \frac{\log x}{(x^2+1)^2}\,dx$



verify and use:

$$\mathrm{Re}\int_{\gamma_2} f(z)\,dz = \mathrm{Re}\int_{\gamma_1} f(z)\,dz$$

(and each limit to the integral we wish to find.)

Plus, the imaginary part of this computation will give you the value of

$$\int_0^\infty \frac{1}{\left(x^2+1\right)^2}\,\mathrm{d}x$$